Math 111 (en)

Lectures : $2 \times 2h$ / week

Exercise sessions : $2h$

We use Moodle for (almost) everything

Schedule :

- Monday : post homework + mini-solution
- Thursday : work on homework in exercise session
- Friday : post solution

Grading : 100 % final exam in winter session

↳ we will also have 3 graded homeworks +
+ 1 mock exam in the second half of the course (these will not affect your final grade, but the point is to get feedback from us)

Please ask any question, mathematical or logistical, on the Ed Forum on Moodle

Linear algebra   Literature studies

linear transformations — ideas, emotions

using as language

matrices — words

Proofs are important to further your understanding (often the logical process is more important than the answer)

<u>Goal</u>: study systems of linear equations

$$\begin{cases} X + Y = 1 \\ X - Y = 7 \end{cases}$$

unknowns $\in \mathbb{R}$

"belongs" "set of real numbers"

<u>How to solve</u>:

$$\begin{cases} X + Y = 1 \\ X - Y = 7 \end{cases}$$

(+)

$X + X + Y + (-Y) = 8$

$2x = 8 \xRightarrow{\cdot \frac{1}{2}} x = 4$

"implies"

$Y = 1 - x = 1 - 4 = -3$

back substitution

$$\begin{cases} X + Y = 1 \\ X - Y = 7 \end{cases}$$

(−)

$X - X + Y - (-Y) = 1 - 7$

$2Y = -6 \xRightarrow{\cdot 1/2} Y = -3$

$X = 1 - Y = 1 - (-3) = 4$

back substitution

in both ways of solving, we get the solution $(x, Y) = (4, -3)$

take $m, n \in \mathbb{N}$ "set of natural numbers"

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots\cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots\cdots + a_{2n}x_n = b_2 \\ \vdots \qquad\qquad\qquad\qquad\qquad\qquad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots\cdots + a_{mn}x_n = b_m \end{cases} \quad \Bigg\} \; m \text{ equations}$$

n unknowns

real numbers:
- a_{ij} are fixed, known numbers called coefficients
- x_j are unknown numbers called variables
- b_i are known real numbers called "RHS", right-hand side

A matrix is a $m \times n$ table (rows, columns)

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} = A$$

which encodes the coefficients

The augmented matrix (of a system) is

$$\left(\begin{array}{c|c} \cdots\cdots\cdots & \begin{matrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{matrix} \end{array}\right) = (A \mid b)$$

Example

$$\begin{cases} 1\cdot X + 1\cdot Y = 1 \\ 1\cdot X + (-1)Y = 7 \end{cases} \qquad A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \qquad (A \mid b) = \left(\begin{array}{cc|c} 1 & 1 & 1 \\ 1 & -1 & 7 \end{array}\right)$$

Goal: solve any $\underbrace{m}_{\#\text{ equations}} \times \underbrace{n}_{\#\text{ unknowns}}$ system of equations

i.e. find all collections of numbers $(x_1, \dots, x_n)$ which simultaneously solve all equations in your system

A system can have:
- 0 solutions
- 1 solutions
- ∞ solutions

How to solve: transform the system through (elementary) row operations

1. swap any two equations
$$\left(\begin{array}{ccc|c} & \vdots & & \vdots \\ a_{i1} & \cdots & a_{in} & b_i \\ & \vdots & & \vdots \\ a_{j1} & \cdots & a_{jn} & b_j \\ & \vdots & & \vdots \end{array}\right) \leftrightarrow \left(\begin{array}{ccc|c} & \vdots & & \vdots \\ a_{j1} & \cdots & a_{jn} & b_j \\ & \vdots & & \vdots \\ a_{i1} & \cdots & a_{in} & b_i \\ & \vdots & & \vdots \end{array}\right)$$

2. multiply any equation by any $\lambda \neq 0$
$$\left(\begin{array}{ccc|c} & \vdots & & \vdots \\ a_{i1} & \cdots & a_{in} & b_i \\ & \vdots & & \vdots \end{array}\right) \leftrightarrow \left(\begin{array}{ccc|c} & \vdots & & \vdots \\ \lambda a_{i1} & \cdots & \lambda a_{in} & \lambda b_i \\ & \vdots & & \vdots \end{array}\right)$$

3. add any multiple of any row to any other row
$$\left(\begin{array}{ccc|c} & \vdots & & \vdots \\ a_{i1} & \cdots & a_{in} & b_i \\ & \vdots & & \vdots \\ a_{j1} & \cdots & a_{jn} & b_j \\ & \vdots & & \vdots \end{array}\right) \leftrightarrow \left(\begin{array}{ccc|c} a_{i1} & \cdots & a_{in} & b_i \\ a_{j1}+\lambda a_{i1} & \cdots & a_{jn}+\lambda a_{in} & b_j+\lambda b_i \end{array}\right)$$

Example:

$$\begin{cases} 1\cdot x_1 - 2x_2 + x_3 = 0 \\ \quad\quad 2x_2 - 8x_3 = 8 \\ 5x_1 \quad\quad - 5x_3 = 10 \end{cases} \qquad \left(\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 5 & 0 & -5 & 10 \end{array}\right)$$

row 3 $\rightsquigarrow$ row 3 $-\ 5 \cdot$ row 1

$$\begin{cases} 1\cdot x_1 - 2x_2 + x_3 = 0 \\ \quad\quad 2x_2 - 8x_3 = 8 \\ \quad\quad 10x_2 - 10x_3 = 10 \end{cases} \qquad \left(\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & 10 & -10 & 10 \end{array}\right)$$

row 3 $\rightsquigarrow$ row 3 $-\ 5 \cdot$ row 2

$$\begin{cases} 1\cdot x_1 - 2x_2 + x_3 = 0 \\ \quad\quad 2x_2 - 8x_3 = 8 \end{cases} \qquad \left(\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \end{array}\right)$$

$$30x_3 = -30 \qquad \left(\begin{array}{ccc|c} 0 & 0 & 30 & -30 \end{array}\right)$$

(row) echelon form

row 3 $\rightsquigarrow \frac{\text{row } 3}{30}$, row 2 $\rightsquigarrow \frac{\text{row } 2}{2}$

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ \qquad x_2 - 4x_3 = 4 \\ \qquad\qquad x_3 = -1 \end{cases} \qquad \left(\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & -1 \end{array}\right)$$

row 2 $\rightsquigarrow$ row 2 + 4 · row 3

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ \qquad x_2 \qquad = 0 \\ \qquad\qquad x_3 = -1 \end{cases} \qquad \left(\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{array}\right)$$

row 1 $\rightsquigarrow$ row 1 + 2 · row 2 + (−1) · row 3

$$\begin{cases} x_1 \qquad\qquad = 1 \\ \qquad x_2 \qquad = 0 \\ \qquad\qquad x_3 = -1 \end{cases} \qquad \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{array}\right)$$

unique solution is $(x_1, x_2, x_3) = (1, 0, -1)$

reduced (row) echelon form

DEF 1.1: a matrix is in (row) echelon form if both

- any all-zero row lies at the very bottom of matrix
- the pivots of the non-zero rows go strictly

to the right as we go down

$$(A|b) = \left(\begin{array}{cccc|c} \boxed{3} & 7 & 5 & 9 & 17 \\ 0 & 0 & \boxed{2} & -3 & 8 \\ 0 & 0 & 0 & 0 & 3 \end{array}\right)$$

all-zero row

sometimes called "leading entry"

pivot = left-most non zero entry of some non-zero row of A

not in echelon form:

$$\left(\begin{array}{ccccc|c} 0 & 0 & \boxed{3} & 19 & \frac{7}{2} & 0 \\ \boxed{7} & \pi & e & 2.9 & 17 & 0 \\ 0 & 0 & 0 & \boxed{-2} & -1 & 2 \end{array}\right)$$

not in echelon form:

$$\left(\begin{array}{ccccc|c} 0 & 0 & \boxed{3} & 19 & \frac{7}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \boxed{-2} & -1 & 2 \end{array}\right)$$

DEF 1.2: a matrix is called reduced (row) echelon form if

- it is in echelon form
- all of its pivots (leading entries) are 1
- any coefficient above a pivot is 0

$$\left(\begin{array}{ccccccccc|c} 0 & \boxed{1} & -1 & 6 & 5 & 0 & 0 & 4 & \frac{7}{2} & 7 \\ 0 & 0 & 0 & 0 & 0 & \boxed{1} & 0 & e & \pi & 19 \\ 0 & 0 & 0 & 0 & 0 & 0 & \boxed{1} & 9 & 17 & -3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \end{array}\right)$$

is in reduced echelon form (REF)

How to solve a system with the matrix in REF

Ex:
$$\begin{array}{c} \left(\begin{array}{ccc|c} \boxed{1} & 0 & 6 & 19 \\ 0 & \boxed{1} & \frac{1}{2} & 1 \\ 0 & 0 & 0 & 0 \end{array}\right) \\ \text{pivot columns} \qquad \text{free column} \end{array}$$

$$\begin{cases} x_1 \qquad + 6x_3 = 19 \\ \qquad x_2 + \frac{x_3}{2} = 1 \\ \qquad\qquad 0 = 0 \ \text{(trivial equation)} \end{cases}$$

pivot variables

free variable, i.e. x_3 can be anything, e.g. $x_3 = t$

solution is
$$(x_1, x_2, x_3) = \left(19 - 6t, 1 - \frac{t}{2}, t\right), \ \forall t \in \mathbb{R}$$

Ex:
$$\begin{array}{c} \left(\begin{array}{cccc|c} \boxed{1} & 7 & 0 & 6 & -2 \\ 0 & 0 & \boxed{1} & 5 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \\ \text{pivot columns} \qquad \text{free columns} \end{array}$$

$$\begin{cases} x_1 + 7x_2 \qquad + 6x_4 = -2 \\ \qquad\qquad x_3 + 5x_4 = 3 \\ \qquad\qquad\qquad 0 = 0 \end{cases}$$

pivot variables free variables

the free variables could be anything, say $x_2 = t$ and $x_4 = s$

then the equations determine the pivot variables

$$x_1 = -2 - 7x_2 - 6x_4 = -2 - 7t - 6s$$
$$x_3 = 3 - 5x_4 = 3 - 5s$$

$$\Downarrow$$

$x_1 = -2 - 7t - 6s$

$x_3 = 3 - 5s$

$\Rightarrow$ solution is $(x_1, x_2, x_3, x_4) = (-2-7t-6s, t, 3-5s, s)$ $\forall s, t \in \mathbb{R}$

Note : if some all-zero row corresponds to the equation

$$0 \cdot x_1 + \dots + 0 \cdot x_n = b$$

and $b \neq 0$, then the system has no solutions.